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A BASIS ON THE SPACE OF WHITTAKER FUNCTIONS FOR THE REPRESENTATIONS OF THE DISCRETE SERIES - THE CASE OF $Sp(2;\mathbb{R})$ -

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We investigate Whittaker functions of the discrete series of the real symplectic group $Sp(2;\mathbb{R})$. We determine a basis on the space of Whittaker functions and find integral expressions of their functions by classical special functions.

1. POWER SERIES SOLUTION

We consider the following system of diffeential equations for $\kappa_1, \kappa_2, \mu, \nu$ in $\mathbb{C}$:

$$\{\partial_1\partial_2 - \kappa_1(a_1/a_2)^2\}\phi(a_1,a_2) = 0, \tag{1.1}$$

$$\{(\partial_1+\partial_2)^2 + 2\mu(\partial_1+\partial_2) + \mu^2 - \nu^2 + 2\kappa_2 a_2^2\partial_2\}\phi(a_1,a_2) = 0. \tag{1.2}$$

This system has power series solutions for $(\frac{a_1}{a_2}, a_2^2)$ in a neighborhood of the origin. For ρ_1, ρ_2 in $\mathbb{C}$, we define the formal power series $\phi_{\rho_1,\rho_2}(a_1,a_2)$ by

$$\phi_{\rho_1,\rho_2}(a_1,a_2) = \left(\frac{a_1}{a_2}\right)^{\rho_1} {a_2}^{\rho_2} \sum_{m,n=0}^{\infty} c_{m,n} \left(\frac{a_1}{a_2}\right)^m a_2^n, \tag{1.3}$$

We assume $c_{0,0} \neq 0$ and ϕ_{ρ_1,ρ_2} satisfies the system $(1.1), (1.2)$. Then we have the following result:

Proposition 1.1. *We put for any fixed $c \neq 0$ in $\mathbb{C}$,*

$$c_{m,n} = \begin{cases} 0, & \text{if } m \text{ or } n \text{ is odd,} \\ c\left(-\dfrac{\kappa_1}{4}\right)^k {\kappa_2}^l \dfrac{1}{\Gamma\left(\frac{\rho_1}{2}+k+1\right)\Gamma\left(\frac{\rho_1-\rho_2}{2}+k-l+1\right)} & \\ \qquad \times \dfrac{1}{\Gamma\left(\frac{\rho_2+\mu+\nu}{2}+l+1\right)\Gamma\left(\frac{\rho_2+\mu+\nu}{2}+l+1\right)}, & \\ & \text{if } (m,n) = (2k,2l) \in 2\mathbb{Z}\times 2\mathbb{Z}. \end{cases}$$

Then for each (ρ_1,ρ_2) in $\{(0,-\mu\pm\nu),(-\mu\pm\nu,-\mu\pm\nu)\}$, ϕ_{ρ_1,ρ_2} given in (1.3) is absolutely convergent for any $\kappa_1,\kappa_2,\mu,\nu$ in $\mathbb{C}$, in all $\left(\dfrac{a_1}{a_2}, {a_2}^2\right)$ in $\mathbb{C}\times\mathbb{C}$, and a solution of the system $(1.1), (1.2)$.

Here if $\kappa_1 = 0$ (*resp.* $\kappa_2 = 0$), we put $\kappa_1{}^0$ (*resp.* $\kappa_2{}^0$) $= 1$.
For (κ_1, κ_2) in $\mathbb{C}^2$ such that $\kappa_1\kappa_2 = 0$, Proposotion(1.1) means the following result:

Corollary 1.1. *The system of differential equations* (1.1), (1.2) *has the following four solutions* $f_{i,j}$ $(i,j = 0,1)$ *for three cases:*

(1) *if* $\kappa_1 = \kappa_2 = 0$, $f_{i,j}(a_1, a_2) = \left(\frac{a_1}{a_2}\right)^{i\{-\mu+(-1)^j\nu\}} a_2^{-\mu+(-1)^j\nu}$,

(2) *if* $\kappa_1 = 0$ *and* $\kappa_2 \neq 0$, $f_{i,j}(a_1, a_2) = \left(\frac{a_1}{a_2}\right)^{i\{-\mu+(-1)^j\nu\}} I_{(-1)^j\nu}(2\sqrt{\kappa_2}a_2)$,

(3) *if* $\kappa_1 \neq 0$ *and* $\kappa_2 = 0$,

$$f_{i,j}(a_1, a_2) = (a_1a_2)^{\frac{1}{2}\{-\mu+(-1)^j\nu\}} I_{(-1)^i\{-\frac{1}{2}(-\mu+(-1)^j\nu)-k\}}\left(\frac{\sqrt{-\kappa_1}a_1}{a_2}\right),$$

where we denote by $I_\nu(z)$ *the modified Bessel function:*

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\ \Gamma(\nu+k+1)}, \quad \textit{for } |\arg(z)| < \pi.$$

For the case $\kappa_1\kappa_2 \neq 0$, we have the following expressions of the power series solutions ϕ_{ρ_1,ρ_2}. :

Definition 1.1. We define for $i,j = 0,1$, $\left|\arg\left(\sqrt{-\kappa_1}\frac{a_1}{a_2}\right)\right| < \pi$,

$$f_{i,j}(a_1, a_2) =$$
$$\frac{2\pi\sqrt{-1}}{4^\mu}\sum_{k=0}^{\infty}\frac{(\sqrt{-\kappa_1\kappa_2}a_1a_2/2)^{\frac{1}{2}\{-\mu+(-1)^j\nu\}+k}}{k!\ \Gamma((-1)^j\nu+k+1)} I_{(-1)^i\{-\frac{1}{2}(-\mu+(-1)^j\nu)-k\}}\left(\frac{\sqrt{-\kappa_1}a_1}{a_2}\right),$$

and for each $(\rho_1, \rho_2) \in \{(0, -\mu\pm\nu), (-\mu\pm\nu, -\mu\pm\nu)\}$,

$$\tilde{\phi}_{\rho_1,\rho_2} = \frac{2\pi\sqrt{-1}}{4^\mu}\left(\frac{-\kappa_1}{4}\right)^{\frac{\rho_1}{2}} \kappa_2{}^{\frac{\rho_2}{2}}\ \phi_{\rho_1,\rho_2}\ .$$

Then we have the following result:

Theorem 1.1. (1) *There are the following relations between* $\{f_{i,j}|i,j = 0,1\}$ *and* $\{\phi_{\rho_1,\rho_2}|(\rho_1,\rho_2) = (0,-\mu\pm\nu), (-\mu\pm\nu, -\mu\pm\nu)\}$*:*

$$\tilde{\phi}_{\rho_1,\rho_2} = \begin{cases} f_{0,0}\ , & \textit{if } (\rho_1,\rho_2) = (0, -\mu+\nu)\ , \\ f_{0,1}\ , & \textit{if } (\rho_1,\rho_2) = (0, -\mu-\nu)\ , \\ f_{1,0}\ , & \textit{if } (\rho_1,\rho_2) = (-\mu+\nu, -\mu+\nu)\ , \\ f_{1,1}\ , & \textit{if } (\rho_1,\rho_2) = (-\mu-\nu, -\mu-\nu)\ . \end{cases}$$

(2) *For each* (i,j), $f_{i,j}$ *has the following integral formula:*

$$f_{i,j}(a_1,a_2)=\int_{(-1)^i C_i} t^{-\frac{1}{2}\{\mu+2\}} I_{(-1)^j\nu}\left(\frac{\sqrt{t}}{2}\right)\exp\left(\frac{t}{16\kappa_2 a_2{}^2}-\frac{4\kappa_1\kappa_2 a_1{}^2}{t}\right)\,dt$$

Here we denote by C_0 *and* C_1 *the following contour:*

$$C_0=\{-16\kappa_2 a_2^2 z \mid z\in C\},$$
$$C_1=\left\{\frac{4\kappa_1\kappa_2 a_1^2}{z}\,\middle|\, z\in C\right\},$$

where C *is the contour which starts from a point* $+\infty$ *on the real axis,proceeds along the realaxis to* 1 , *describes a circle counter-clockwise round the origin and returns to* $+\infty$ *along the real axis.*

By Theorem(1.1), we know when ϕ_{ρ_1,ρ_2}, $(\rho_1,\rho_2)=(0,-\mu\pm\nu),(-\mu\pm\nu,-\mu\pm\nu)$ are linearly independent.

Corollary 1.2. *If and only if both* ν, $\frac{-\mu+\nu}{2}$ *and* $\frac{-\mu-\nu}{2}$ *are not in* $\mathbb{Z}$, *the set* $\{\phi_{\rho_1,\rho_2}|\,(\rho_1,\rho_2)=(0,-\mu\pm\nu),(-\mu\pm\nu,-\mu\pm\nu)\}$ *is a basis on the space of solutions for the system* (1.1), (1.2).

2. ANOTHER BASIS ON THE SPACE OF SOLUTIONS

The basis $\{f_{i,j}\,|\,i,j=0,1\}$ does not contain a moderate growth function on $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$. Here $\mathbb{R}_{>0}$ denotes the set of positive element in $\mathbb{R}$. Now we construct another basis which contains a moderate growth function on $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$.

Definition 2.1. We set for each $l=0,1$,

$$f_l=\begin{cases}\dfrac{1}{2\sqrt{-1}}\dfrac{(-1)^{\frac{1}{2}\{-\mu+(-1)^l\nu\}}(f_{1,l}-f_{0,l})}{\sin\{-\frac{1}{2}(-\mu+(-1)^l\nu)\pi\}}, & \text{if } \dfrac{1}{2}\{-\mu+(-1)^l\nu\}\notin\mathbb{Z},\\[2ex] \lim\limits_{\frac{1}{2}\{-\mu+(-1)^l\nu\}\to m}\dfrac{1}{2\sqrt{-1}}\dfrac{(-1)^{\frac{1}{2}\{-\mu+(-1)^l\nu\}}(f_{1,l}-f_{0,l})}{\sin\{-\frac{1}{2}(-\mu+(-1)^l\nu)\pi\}}, & \text{if } \dfrac{1}{2}\{-\mu+(-1)^l\nu\}=m\in\mathbb{Z},\end{cases}$$

$\phi_1=f_{0,0}$, $\qquad\phi_2=f_0$,

$$\phi_3\ (\mathit{resp.}\ \phi_4)=\begin{cases}\dfrac{\pi}{2}\dfrac{f_{0,1}-f_{0,0}}{\sin\nu\pi}\left(\mathit{resp.}\ \dfrac{\pi}{2}\dfrac{f_1-f_0}{\sin\nu\pi}\right), & \text{if } \nu\notin\mathbb{Z},\\[2ex] \lim\limits_{\nu\to m}\dfrac{\pi}{2}\dfrac{f_{0,1}-f_{0,0}}{\sin\nu\pi}\left(\mathit{resp.}\ \lim\limits_{\nu\to m}\dfrac{\pi}{2}\dfrac{f_1-f_0}{\sin\nu\pi}\right), & \text{if } \nu=m\in\mathbb{Z},\end{cases}$$

Then we have the following:

Theorem 2.1. *For any $\kappa_1, \kappa_2, \mu, \nu \in \mathbb{C}$, the set $\{\phi_i \mid i = 1,2,3 \text{ or } 4\}$ is a basis on the space of solutions for the system (1.1), (1.2). Moreover we have the following integral formula of ϕ_3:*

$$\phi_3(a_1, a_2) = \int_{C_0} t^{-\frac{1}{2}\mu} K_\nu\left(\frac{\sqrt{t}}{2}\right) \exp\left(\frac{t}{16\kappa_2 a_2^2} - \frac{4\kappa_1\kappa_2 a_1^2}{t}\right)\frac{dt}{t},$$

and when $\left|\arg\left(\frac{\sqrt{-\kappa_1}a_1}{a_2}\right)\right| < \frac{\pi}{4}$, we have the following integral formula of ϕ_2 and ϕ_4:

$$\phi_2(a_1, a_2) = \int_0^{(-16\kappa_2 a_2^2)\cdot\infty} t^{-\frac{1}{2}\mu} I_\nu\left(\frac{\sqrt{t}}{2}\right) \exp\left(\frac{t}{16\kappa_2 a_2^2} - \frac{4\kappa_1\kappa_2 a_1^2}{t}\right)\frac{dt}{t},$$

$$\phi_4(a_1, a_2) = \int_0^{(-16\kappa_2 a_2^2)\cdot\infty} t^{-\frac{1}{2}\mu} K_\nu\left(\frac{\sqrt{t}}{2}\right) \exp\left(\frac{t}{16\kappa_2 a_2^2} - \frac{4\kappa_1\kappa_2 a_1^2}{t}\right)\frac{dt}{t}.$$

Here we denote by K_ν the Bessel function:

$$K_\nu(z) = \begin{cases} \dfrac{\pi}{2}\dfrac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, & \text{if } \nu \notin \mathbb{Z}, \\ \lim_{\nu\to m} \dfrac{\pi}{2}\dfrac{I_{-\nu(z)} - I_\nu(z)}{\sin \nu\pi}, & \text{if } \nu = m \in \mathbb{Z}. \end{cases}$$

and $\int_0^{(-16\kappa_2 a_2^2)\cdot\infty} dt$ implies that we exchange the variable s in the usual integral $\int_0^\infty ds$ on $(0,\infty)$ for $s = -16\kappa_2 a_2^2 t$.

Next we shall obtain some evaluations of $|\phi_i(a_1, a_2)|$ $(1 \le i \le 4)$. We need some evaluations of the Bessel functions $I_\nu(z)$ and $K_\nu(z)$:

Lemma 2.1. *We assume that $\nu \in \mathbb{R}$. Then, for any $\epsilon > 0$, there exist constants $C_\epsilon, C'_\epsilon > 0$ such that:*

$$\frac{K_\nu(z)}{\Gamma\left(\delta_\nu + \frac{1}{2}\right)} \le C_\epsilon \left(\frac{z}{2}\right)^{\delta_\nu} \exp(-z), \qquad \text{for } z \in \mathbb{R} \text{ and } z \ge \epsilon,$$

$$\frac{|I_\nu(z)|}{\Gamma\left(\delta_n u + \frac{1}{2}\right)} \le C'_\epsilon \left(\frac{z}{2}\right)^{\delta_\nu} \exp(z), \qquad \text{for } z \in \mathbb{R} \text{ and } z \ge \epsilon.$$

Here for $\nu \in \mathbb{C}$ we denote by δ_ν the following number:

$$\delta_\nu = \begin{cases} \nu, & \text{if } \Re(\nu) > 0, \\ -\nu, & \text{if } \Re(\nu) < 0. \end{cases}$$

We set for $\nu \in \mathbb{R}$, $j = 0, 1$,

$$X_{j,\nu} = \begin{cases} \{k \in \mathbb{N} \mid k \geq |\nu| + 1\}, & \text{if } \nu \in \mathbb{Z} \text{ and } (-1)^j \nu < 0, \\ \mathbb{N}, & \text{otherwise,} \end{cases}$$

$$k_{j,\nu} = \min\{k \in X_{j,\nu}\},$$

$$M_{j,\mu,\nu} = \sup_{l \in X_{j,\nu}} \frac{|\frac{1}{2}(-\mu + (-1)^j \nu + 1) + l|}{|(-1)^j \nu + 1 + l|},$$

$$M_{\mu,\nu} = \max_{j=0,1} M_{j,\mu,\nu}.$$

We denote by $c_j^{\mu,\nu}$ ($j = 0, 1$; $\mu, \nu \in \mathbb{R}$) the following constant:

$$c_{j,\mu,\nu} = \begin{cases} \left|\Gamma\left(\frac{1}{2}(-\mu - (-1)^j \nu + 1)\right)\right|, & \text{if } \nu \in \mathbb{Z} \text{ and } (-1)^j \nu < 0, \\ \dfrac{\left|\Gamma\left(\frac{1}{2}(-\mu + (-1)^j \nu + 1)\right)\right|}{|\Gamma((-1)^j \nu + 1)|}, & \text{otherwise.} \end{cases}$$

For simplicity, we wrirte $c_{i,j} = c_{i,j,\mu,\nu}$, $M_i = M_{i,\mu,\nu}$, $M = M_{\mu,\nu}$ and $k_j = k_{j,\nu}$. Then we obtain the following results of ϕ_i from Lemma(2.1) and Theorem(2.1):

Corollary 2.1. *We assume that $\kappa_1, \kappa_2, \mu, \nu \in \mathbb{R}$, $\kappa_2 \neq 0$, $\kappa_1 < 0$ and $a_1, a_2 > 0$. Then we obtain the following results:*
(1) If $-\mu + \nu$ and $-\mu - \nu$ are not contained in the set $\{x \in 2\mathbb{Z} + 1 \mid x \leq -1\}$, then for any fixed $\epsilon > 0$, we obtain the following evaluations of ϕ_i ($1 \leq i \leq 4$):

$$|\phi_1(a_1, a_2)| \leq \frac{2\pi c_0 C'_\epsilon}{4^\mu} M_0^{-k_0} \left(-\frac{\kappa_1|\kappa_2|}{4} a_1^2\right)^{\frac{1}{2}(-\mu+\nu)} \exp\left(-M_0 \frac{\kappa_1|\kappa_2|}{4} a_1^2 + \sqrt{-\kappa_1} \frac{a_1}{a_2}\right),$$

$$|\phi_2(a_1, a_2)| \leq \frac{2 c_0 C_\epsilon}{4^\mu} M_0^{-k_0} \left(-\frac{\kappa_1|\kappa_2|}{4} a_1^2\right)^{\frac{1}{2}(-\mu+\nu)} \exp\left(-M_0 \frac{\kappa_1|\kappa_2|}{4} a_1^2 - \sqrt{-\kappa_1} \frac{a_1}{a_2}\right),$$

$$|\phi_3(a_1, a_2)| \leq \frac{\pi^2 (c_0 + c_1) C'_\epsilon}{4^\mu} \max_{j=0,1} \left\{ \left(-\frac{\kappa_1|\kappa_2|}{4} a_1^2\right)^{\frac{1}{2}(-\mu+(-1)^j \nu)} M_j^{-k_j} \right\}$$
$$\times \exp\left(-M \frac{\kappa_1|\kappa_2|}{4} a_1^2 + \sqrt{-\kappa_1} \frac{a_1}{a_2}\right),$$

$$|\phi_4(a_1, a_2)| \leq \frac{\pi (c_0 + c_1) C_\epsilon}{4^\mu} \max_{j=0,1} \left\{ \left(-\frac{\kappa_1|\kappa_2|}{4} a_1^2\right)^{\frac{1}{2}(-\mu+(-1)^j \nu)} M_j^{-k_j} \right\}$$
$$\times \exp\left(-M \frac{\kappa_1|\kappa_2|}{4} a_1^2 - \sqrt{-\kappa_1} \frac{a_1}{a_2}\right),$$

$$\textit{for } \frac{a_1}{a_2} \geq \epsilon, a_2 > 0.$$

(2) ϕ_2 (*resp.* ϕ_4) *is positive real valued for any* $\nu > 0$ (*resp.* $\nu \in \mathbb{R}$). *Moreover we assume that* $\kappa_2 < 0$. *Then, for any fixed* $\frac{a_1}{a_2} > 0$, $\phi_3(a_1, a_2)$ *and* $\phi_4(a_1, a_2)$ *are rapidly decreasing as* $a_2 \to +\infty$.

3. WHITTAKER FUNCTIONS FOR THE REPRESENTATIONS OF THE DISCRETE SERIES - THE CASE OF $Sp(2;\mathbb{R})$ -

3.1. Structure of Lie group and Lie algebra. Let G be the symplectic group $Sp(2;\mathbb{R})$ realized as

$$G = \{g \in SL_4(\mathbb{R}) \mid {}^t gJg = J\}, \quad \text{with } J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \in M_4(\mathbb{R}),$$

where ${}^t g$ denotes the transpose of a matrix g and 1_2 denotes a unit matrix of size 2.

Let $O(4)$ be the orthogonal group of degree 2. Take a maximal compact subgroup $K = G \cap O(4)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebra of G, K, respectively. Let $\theta(X) = -{}^t X$ be a Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$.

We set $\mathfrak{a} = \mathbb{R}H_1 + \mathbb{R}H_2$ with $H_1 = diag(1,0,-1,0), H_2 = diag(0,1,0,-1)$. Then $\mathfrak{a}$ is a maximally Cartan subalgebra of $\mathfrak{g}$ and the restricted root system $\Delta = \Delta(\mathfrak{g};\mathfrak{a})$ is expressed as $\Delta = \Delta(\mathfrak{g};\mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}$, where λ_j is the dual of H_j. We choose a positive root system Δ^+ as $\Delta^+ = \{\lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2\}$. We also denote the corresponding nilpotent subalgebra by $\mathfrak{n} = \sum_{\beta\in\Delta^+} \mathfrak{g}_\beta$. Here $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ corresponding to $\beta \in \Delta^+$. Then one obtains an Iwasawa decomposition of $\mathfrak{g}$ and G ; $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$, $G = NAK$ with $A = \exp\mathfrak{a}$, $N = \exp\mathfrak{n}$.

3.2. Representation of the maximal compact subgroup. Firstly, we review the parametrization of the finite-dimensional irreducible representations of $SL_2(\mathbb{C})$. Let $\{f_1, f_2\}$ be the standard basis of the vector space $V = V_1 = \mathbb{C}\oplus\mathbb{C}$. Then $GL_2(\mathbb{C})$ acts on V by matrix multiplication. We denote the symmetric tensor space of 2 dimension by $V_d = S^d(V)$. Here $V_0 = \mathbb{C}$. We consider V_d as a $SL_2(\mathbb{C})$-module by

$$sym^d(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_d.$$

It is well known that all the finite-dimensional irreducible (polynomial) representations of $SL_2(\mathbb{C})$ can be obtained in this way. By Weyl's unitary trick, all irreducible unitary representations of $SU(2)$ are obtained by restriction of sym^d $(d \geq 0)$.

The maximal compact subgroup K is isomorphic to the unitary group $U(2)$ of degree 2 by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \to A + \sqrt{-1}B, \quad \text{for } \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K.$$

For $d, m \in \mathbb{Z}, d \geq 0$, we define a holomorphic representation $(\sigma_{d,m}, V_d)$ of $GL_2(\mathbb{C})$ by $\sigma_{d,m}(g) = sym^d(g) \otimes \det(g)^m$. Then we know $\hat{U(2)} = \{\sigma_{d,m}|_{U(2)} \mid d, m \in \mathbb{Z}, d \geq 0\}$. We set $\lambda = (\lambda_1, \lambda_2) = (m + d, m)$ and $\tau_\lambda = \sigma_{d,m}|_{U(2)}$. By the isomorphism between

K and $U(2)$, we obtain $\hat{K} = \{(\tau_\lambda, V_\lambda) \mid \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}, \lambda_1 \geq \lambda_2\}$. We choose the basis of V_λ as

$$V_\lambda = \left\{ v_k = \frac{n!}{k!(n-k)!} f_1^{\otimes k} \otimes f_2^{\otimes(n-k)} \text{ (symmetric tensor)} \mid 0 \leq k \leq n \right\}_{\mathbb{C}}.$$

3.3. Characters of the unipotent radical. The commutator subgroup $[N, N]$ of N is given by

$$[N,N] = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & 0 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \;\middle|\; n_1, n_2 \in \mathbb{R} \right\}.$$

Hence a unitary character η of N is written for some constant $\eta_0, \eta_3 \in \mathbb{R}$ as

$$\left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \mapsto \exp\{\sqrt{-1}(\eta_0 n_0 + \eta_3 n_3)\} \in \mathbb{C}^\times.$$

A unitary character η of N is said to be non-degenerate if $\eta_0\eta_3 \neq 0$.

3.4. Parametrization of the discrete series. Let us now parametrize the discrete series of $Sp(2;\mathbb{R})$. Take a compact Cartan subalgebra $\mathfrak{h}$ defined by $\mathfrak{h} = \mathbb{R}h_1 \oplus \mathbb{R}h_2$ with $h_1 = X_{13} - X_{31}$, $h_2 = X_{24} - X_{42}$, where the $X_{ij}'s$ are elementary matrices given by $X_{ij} = (\delta_{ip}\delta_{jq})_{1 \leq p,q \leq 4}$, with Kronecker's delta $\delta_{i,p}$, and let $\mathfrak{h}_\mathbb{C}$ be its complexification. Then the absolute root system is expressed as

$$\tilde{\Delta} = \Delta(\mathfrak{g}; \mathfrak{h}) = \{\pm(2,0), \pm(0,2), \pm(1,1), \pm(1,-1)\},$$

where by $\beta = (r,s)$, we mean $r = \beta(-\sqrt{-1}h_1), s = \beta(-\sqrt{-1}h_2)$. Let

$$\tilde{\Delta}^+ = \{(2,0), (0,2), (1,1)(1,-1)\}.$$

We write the set of compact positive roots by $\tilde{\Delta}_c^+ = \{(1,-1)\}$. Then there are 4 sets of positive roots $\tilde{\Delta}_J^+$ $(J = I, II, III, IV)$ of $(\mathfrak{g}, \mathfrak{h})$ containing $\Delta_c^+(\mathfrak{g}; \mathfrak{h})$ as follows:

$$\tilde{\Delta}_I^+ = \{(2,0), (1,1), (0,2), (1,-1)\}, \quad \tilde{\Delta}_{II}^+ = \{(1,1), (2,0), (1,-1), (0,-2)\},$$

$$\tilde{\Delta}_{III}^+ = \{(2,0), (1,-1), (0,-2), (-1,-1)\}, \quad \tilde{\Delta}_{IV}^+ = \{(1,-1), (0,-2), (-1,-1), (-2,0)\}.$$

We put $\delta_{G,J} = 2^{-1}\sum_{\beta \in \tilde{\Delta}_J^+} \beta$ (resp. $\delta_K = 2^{-1}\sum_{\beta \in \tilde{\Delta}_c^+} \beta$), the half sum of positive roots (resp. the half sum of compact positive roots). By definition, the space of Harish-Chandra parameters Ξ_c^+ is given by

$$\Xi_c^+ = \{\Lambda \in \mathfrak{h}_\mathbb{C}^* \mid \Lambda + \delta_{G,I} \text{ is analytically integral and } \Lambda \text{ is regular and } \tilde{\Delta}^+\text{-dominant}\}.$$

For each $J = I, II, III, IV$, we set $\Xi_J = \{\Lambda \in \Xi_c^+ \mid \langle \Lambda, \alpha \rangle > 0 \ (\alpha \in \tilde{\Delta}_J^+)\}$. Then Ξ_c^+ is written as a disjoint union $\Xi_c^+ = \coprod_{J=I}^{IV} \Xi_J$.

It is well-known that there exists a bijection from Ξ_c^+ to the set of equivalence classes of discrete series representations of G. Let π_Λ be the discrete series representation associated to Λ in Ξ_J^+ , then τ_λ $(\lambda = \Lambda + \delta_{G,J} - 2\delta_K)$ is the unique minimal K-type of π_Λ. We note that for each Λ in Ξ_c^+, $\lambda = \Lambda + \delta_{G,J} - 2\delta_K$ is called the Blattner parameter. An easy computation implies

$$\Xi_c^+ = \{(\Lambda_1, \Lambda_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid \Lambda_1 \neq 0, \Lambda_2 \neq 0, \Lambda_2 < \Lambda_1, \Lambda_1 + \Lambda_2 \neq 0\}.$$

We note that Ξ_{I} $(resp. \Xi_{IV})$ corresponds to the holomorphic (resp. anti-holomorphic) discrete series, and Ξ_{II} and Ξ_{III} coresponds to the large discrete series in the sence of Vogan,[V].

3.5. Characterization of the minimal K-type of a discrete series representation. Let η be a unitary character of N. Then we set

$$C_\eta^\infty(N \setminus G) = \{\phi : G \to \mathbb{C},\ C^\infty\text{-}class \mid \phi(ng) = \eta(n)\phi(g),\ (n, g) \in N \times G\}.$$

By the right regular action of G, $C_\eta^\infty(N \setminus G)$ has a structure of smooth G-module. For any finite dimensional K-module (τ, V), we set

$$\begin{aligned} &C_{\eta,\tau}^\infty(N \setminus G/K) = \\ &\quad \{F : G \to V,\ C^\infty\text{-}class \mid F(ngk^{-1}) = \eta(n)\tau(k)F(g),\ (n, g, k) \in N \times G \times K\}. \end{aligned}$$

Let (π_Λ, H) be the discrete series representation of G with Harish-Chandra parameter Λ in Ξ_J, $(J = I, II, III, IV)$, and denote its associated $(\mathfrak{g}_\mathbb{C}, K)$-module by the same symbol. For W in $Hom_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda^*, C_\eta^\infty(N \setminus G))$, we define F_W in $C_{\eta,\tau_\lambda}^\infty(N \setminus G/K)$ by

$$W(v^*)(g) = \langle v^*, F_W(g)\rangle, \quad (v^* \in V_\lambda^*, g \in G).$$

Here $(\tau_\lambda, V_\lambda)$ denotes the minimal K-type of π_Λ and $\langle *, * \rangle$ denotes the canonical pairing on $V_\lambda^* \times V_\lambda$.

Now let us recall the definition of the Schmid-operater. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and $Ad = Ad_{\mathfrak{p}_\mathbb{C}}$ be the adjoint representation of K on $\mathfrak{p}_\mathbb{C}$. Then we can define a differential operator $\nabla_{\eta,\lambda}$ from $C_{\eta,\tau_\lambda}^\infty(N \setminus G/K)$ to $C_{\eta,\tau_\lambda \otimes Ad}^\infty(N \setminus G/K)$ as $\nabla_{\eta,\lambda}F = \sum_i R_{X_i}F(\cdot) \otimes X_i$. Here the set $\{X_i\}_i$ is any fixed orthonormal basis of p with respect to the Klilling form on $\mathfrak{g}$ and $R_X F$ denotes the right differential of the function F by X in $\mathfrak{g}$ i.e. $R_X F(g) = \frac{d}{dt}F(g \cdot \exp tX)\Big|_{t=0}$. This operator $\nabla_{\eta,\lambda}$ is called the Schmid operator.

Let $(\tau_\lambda^-, V_\lambda^-)$ be the sum of irreducible K-submodules of $V_\lambda \otimes p_\mathbb{C}$ with heighest weight of the form $\lambda - \beta$ $(\beta \in \tilde{\Delta}_{J,n}^+,\ J = I, II, III, IV)$. Let P_λ be the projection from $V_\lambda \otimes p_\mathbb{C}$ to V_λ^-. We define a differential operator from $C_{\eta,\tau_\lambda}^\infty(N \setminus G/K)$ to $C_{\eta,\tau_\lambda^-}^\infty(N \setminus G/K)$ by $\mathcal{D}_{\eta,\lambda}F(g) = P_\lambda(\nabla_{\eta,\lambda}F(g))$ for $F \in C_{\eta,\tau_\lambda}^\infty(N \setminus G/K)$, $g \in G$. We have the following:

Proposition 3.1 ([Y1] H.Yamashita, Proposition(2.1)). *Let π_Λ be a representation of discrete series with Harish- Chandra parameter $\Lambda \in \Xi_J$ of $Sp(2;\mathbb{R})$. Set $\lambda = \Lambda + \delta_G - 2\delta_K$. Then the linear map*

$$W \in Hom_{\mathfrak{g}_\mathbb{C},K}(\pi_\Lambda^*, C_\eta^\infty(N \setminus G)) \to F_W \in Ker(\mathcal{D}_{\eta,\lambda})$$

is injective, and if Λ is far from the walls of the Wyel chambers, it is bijective.

3.6. A basis on the Whittaker space on $Sp(2;\mathbb{R})$. By the result of Kostant [Ko], and Vogan [V], if η is non-degenerate, we obtain

$$dim_\mathbb{C}\mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda, C_\eta^\infty(N\backslash G)) = \begin{cases} 4, & \text{if } \Lambda \in \Xi_{II} \cup \Xi_{III}, \\ 0, & \text{if } \Lambda \in \Xi_I \cup \Xi_{IV}. \end{cases}$$

Oda proved the following:

Theorem 3.1 ([O] Oda). *Let us assume that η is non-degenerate and $\Lambda \in \Xi_{II}$. We choose the basis $V_\lambda = \{v_k \mid 0 \leq k \leq d\}_\mathbb{C}$ defined in §4.2. Here $d = \lambda_1 - \lambda_2$. Then*
(1) $F \in Ker\mathcal{D}_{\eta,\lambda}$ if and only if F satisfies the following conditions:

$$(\partial_1 - k)h_{d-k} + \sqrt{-1}\eta_0 h_{d-k-1} = 0, \quad k = 0,1,\ldots,d-1,$$

$$\{\partial_1\partial_2 + (a_1/a_2)^2\eta_0^2\}h_d = 0, \tag{3.1}$$

$$\{(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 1)(\partial_1 + \partial_2) - 2\lambda_2 + 1 + 4\eta_3 a_2^2\partial_2\}h_d = 0. \tag{3.2}$$

Here $\partial_i = \frac{\partial}{\partial a_i}$, $i = 1,2$ and $\{h_k \mid 0 \leq k \leq d\}$ is determined by

$$F|_A(a) = \sum_{k=0}^{d} c_k(a)v_k,$$

$$c_k(a) = a_1^{\lambda_2+1}a_2^{\lambda_1}\left(\frac{a_1}{a_2}\right)^k \exp(\eta_3 a_2^2)h_k(a), \quad (a \in A;\ k = 0,1,\cdots,d).$$

(2) If $\eta_3 < 0$, $Ker\mathcal{D}_{\eta,\lambda}$ contains the function F such that h_d has an integral representation:

$$h_d(a) = \int_0^\infty t^{-\lambda_2+\frac{1}{2}}W_{0,-\lambda_2}(t)\exp\left(\frac{t^2}{32\eta_3 a_2^2} + \frac{8\eta_0^2\eta_3 a_1^2}{t^2}\right)\frac{dt}{t}.$$

By Theorem 3.1 , Oda showed that if $\Lambda \in \Xi_{II} \cup \Xi_{III}$ and η is non-degenerate,

$$Hom_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\Lambda^*, \mathcal{A}_\eta(N\backslash G)) \cong \begin{cases} \mathbb{C}, & \eta_3 < 0, \\ 0, & \eta_3 > 0. \end{cases}$$

Here we put

$$\mathcal{A}_\eta(N\backslash G) = \{F \in C_\eta^\infty(N\backslash G) \mid K\text{-finite and for any } X \in U(\mathfrak{g}_\mathbb{C}) \text{ there exists a constant } C_X > 0 \text{ such that } |F(g)| \leq C_X tr({}^t gg),\ g \in G\ \}$$

and $U(\mathfrak{g}_\mathbb{C})$ denotes the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$.

The system of equations (3.1), (3.2) is coincide with the system (1.1), (1.2) with the parameters $\kappa_1 = \eta_0^2$, $\kappa_2 = -2\sqrt{-1}\eta_3$, $\mu = \lambda_2 - 1$, $\nu = -\lambda_2$. And these parameters satisfies the assumptions in the Corollary(2.1). So let us denote by $\phi_i(\kappa_1, \kappa_2, \mu, \nu; a_1, a_2)$ for the function $\phi_i(a_1, a_2)$ $(1 \leq i \leq 4)$ given for $\kappa_1, \kappa_2, \mu, \nu \in \mathbb{C}$ in §3. We set

$$h_d^{(i)}(a_1, a_2) = \phi_i(\eta_0^2, -2\sqrt{-1}\eta_3, \lambda_2 - 1, -\lambda_2; a_1, a_2), \quad \text{for } 1 \leq i \leq 4,\ a_1, a_2 > 0,$$

and determine $h_k^{(i)}$ by the relations

$$(\partial_1 - k)h_{d-k}^{(i)} + \sqrt{-1}\eta_0 h_{d-k-1}^{(i)} = 0, \quad \text{for } 0 \leq k \leq d-1,\ 1 \leq i \leq 4.$$

We define the function $F^{(i)} \in C_\eta^\infty(N\backslash G/K)$ by

$$F^{(i)}|_A(a) = \sum_{0 \leq k \leq d} c_k(a) v_k, \quad \text{with } c_k(a) = a_1^{\lambda_2+1} a_2^{\lambda_1} \left(\frac{a_1}{a_2}\right)^k \exp(\eta_3 a_2^2) h_k(a),$$
$$\text{for } a \in A,\ 0 \leq k \leq d,\ 1 \leq i \leq 4.$$

and set for $t \in \mathbb{C}$, $|\arg t| < \pi$,

$$k_{i,\nu}(t) = \begin{cases} K_\nu(\sqrt{t}/2), & \text{if } i = 1, 2, \\ I_\nu(\sqrt{t}/2), & \text{if } i = 3, 4, \end{cases}$$

Then we obtain the following result:

Theorem 3.2. *Let us assume that η is non-degenerate and $\Lambda \in \Xi_{II}$. Then we obtain the following results:*

(1) *$KerD_{\eta,\lambda}$ has the basis $\{F^{(i)} | 1 \leq i \leq 4\}$ and $h_d^{(i)}$ $(1 \leq i \leq 4)$ have the following integral expressions:*

$$h_d^{(i)}(a) = \int_{C_i} t^{\frac{1}{2}(1-\lambda_2)} k_{i,\nu}(t) \exp\left(\frac{t}{32\eta_3 a_2^2} + \frac{8\eta_0^2\eta_3 a_1^2}{t}\right) \frac{dt}{t}.$$

Here we denote by C_i $(1 \leq i \leq 4)$ the following contour:

$$\int_{C_i} dt = \begin{cases} \int_C dt, & \text{if } i = 1, 3, \\ \int_0^\infty dt, & \text{if } i = 2, 4, \end{cases}$$

where $\int_C dt$ is the contour integral on C given in Theorem (1.1)-(2) and $\int_0^\infty dt$ is the usual integral on $(0, \infty) \subset \mathbb{R}$.

(2) *For any fixed constant $R_1, R_2 > 0$, we denote by D_{R_1,R_2} the domain*

$$D_{R_1,R_2} = \{(a_1, a_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \mid a_1 a_2 \leq R_1 \text{ and } a_1 \leq R_2\}.$$

Then there exist constants $C^{(i)} = C^{(i)}_{R_1,R_2}$ $(1 \le i \le 4)$ *and* $C^{(i)}_k = C^{(i)}_{R_1,R_1,k}$ $(0 \le k \le d;\ i = 1,2)$ *such that*

$$|c^{(i)}_d(a_1,a_2)| \le C^{(i)} a_1^{\lambda_1+1} a_2^{1+m_i\lambda_2} \exp\left((-1)^{i+1}|\eta_0|\frac{a_1}{a_2} + m_i\eta_3 a_2^2\right),$$

$$|c^{(i)}_k(a_1,a_2)| \le C^{(i)}_k a_1^{\lambda_1+1} a_2^{1+m_i\lambda_2} \left(\frac{a_1}{a_2}\right)^{\frac{1-(-1)^{d-k}}{2}} \exp\left((-1)^{i+1}|\eta_0|\frac{a_1}{a_2} + m_i\eta_3 a_2^2\right),$$

$$\textit{for } (a_1,a_2) \in D_{R_1,R_2}.$$

Here we set for $1 \le i \le 4$

$$m_i = \begin{cases} -1, & \textit{if } i = 1,2, \\ \ \ 1, & \textit{if } i = 3,4 \end{cases}$$

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